STB
2019

1. Consider a count variable X following a Poisson distribution with parameter $\theta > 0$, where zero count (i.e., $X = 0$) is not observable. We have n observations $X_1, \ldots, X_n$ from this distribution. Let $\overline{X}$ denote the sample mean.

a) Derive the quantity for which $\overline{X}$ is an unbiased estimator.

b) Suppose that the observed value of $\overline{X}$ is strictly greater than 1. Show that the likelihood function of θ has a unique maximizer.

[5+10]=15

2. Let $\mathcal{P} = \{f_\theta : \theta \in \Theta\}$, where f_θ is a continuous probability density over the support $\mathbb{R}$ for each $\theta \in \Theta$. Suppose that, if X_1, X_2 are independent and identically distributed with density f_θ, then $X_1 + X_2$ is sufficient for θ.

a) Fix $\theta_0 \in \Theta$ and define $s(x, \theta) = \log f_\theta(x) - \log f_{\theta_0}(x) - \log f_\theta(0) + \log f_{\theta_0}(0)$. Prove that

$$s(x_1 + x_2, \theta) = s(x_1, \theta) + s(x_2, \theta), \qquad \text{for all } \theta \in \Theta,\ x_1, x_2 \in \mathbb{R},$$

and hence show that $s(x, \theta) = xs(1, \theta)$ for all x and all θ.

b) Using (a), or otherwise, prove that $\mathcal{P}$ must be an exponential family indexed by θ.

[(5+5)+5]=15

3. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with a common density function $f(x, \theta) = e^{-(x-\theta)} I(x \geq \theta)$, where $\theta \in \mathbb{R}$.

a) Find the maximum likelihood estimator $\widehat{\theta}_n$ of θ based on $X_1, \ldots, X_n$.

b) Show that $\widehat{\theta}_n$ is consistent for θ.

c) For a suitable normalizing factor k_n (to be specified by you), find a non-degenerate limiting distribution of $k_n(\widehat{\theta}_n - \theta)$.

[3+5+7]=15

4. Consider the Gauss-Markov model, $Y = X\beta + \epsilon$, where $\epsilon \sim N_n(0, \sigma^2 I_n)$ and $X_{n \times p}$ has rank $r < p$. Suppose $T_{n \times (p-1)}$ is the matrix formed by the first $p-1$ columns of X and it also has rank r. Let B denote any generalized inverse of $T'T$. Prove that $\widehat{\beta} = \begin{pmatrix} BT'Y \\ 0 \end{pmatrix}$ minimizes $(Y - X\beta)'(Y - X\beta)$.

[15]

5. Suppose $X_1, X_2, \ldots, X_n$ are independent with $X_i \sim N(i\theta, \tau^2)$ for $i = 1, \ldots, n$. Define

$$U = \frac{\sum_{i=1}^n iX_i}{\sum_{i=1}^n i^2}, \qquad V^2 = \frac{\sum_{i=1}^n (X_i - iU)^2}{n-1}.$$

Show that $\frac{1}{\tau^2}(n-1)V^2$ has a chi-square distribution with $(n-1)$ degrees of freedom.

[15]

6. Suppose that $T_1, \ldots, T_n$ are lifetimes of n items started together which are independent and identically distributed having exponential distribution with mean $1/\lambda$. Also let $0 < \tau_1 < \tau_2$ are two prefixed time points when they are observed. At time τ_1 we remove each surviving item, if any, with probability $p \in (0, 1)$, and at time τ_2 we remove all the surviving items, if any, from the study. Instead of observing the T_is, we observe only the four counts as follows:

X_1 = the number of items failed before time τ_1,
X_2 = the number of items removed at time τ_1,
X_3 = the number of items failed between times τ_1 and τ_2,
X_4 = the number of items removed at time τ_2.

a) Obtain the joint distribution of (X_1, X_2, X_3, X_4).

b) Find a maximum likelihood estimate of p based on these four counts.

[9+6]=15

7. A spider and a fly move between locations 1 and 2 at discrete times $1, 2, 3, \ldots$ according to Markov chains with respective transition matrices $\begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix}$ and $\begin{pmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{pmatrix}$. The spider starts from location 1 while the fly starts from location 2. Once they are at the same location, there is no further movement.

a) Find the transition matrix of their joint movement over the following three states:

S_1 = Spider is at location 1 but the fly is at location 2,
S_2 = Spider is at location 2 but the fly is at location 1,
S_3 = Both spider and fly are at the same location.

b) What is the expected time till the two meet at the same location?

[7+8]=15

8. Let $X_1, \ldots, X_n$ be independent and identically distributed having the discrete uniform distribution on $\{1, 2, \ldots, \theta\}$, where $\theta \in \Theta = \{2, 3, 4, 5, \ldots\}$.

a) Given $\theta_0 \in \Theta$ and $0 < \alpha < 1$, find a level-α likelihood ratio test for testing

$$H_0 : \theta \leq \theta_0 \quad \text{against} \quad H_1 : \theta > \theta_0.$$

b) Show that the largest order statistic is not complete.

[6+9]=15